

(3 Hours)

[Total Marks : 100]

28/4/12

Section I

Answer all the 20 objective questions. [40]

1. Let $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be non-zero linear maps such that $\ker(f) \subseteq \ker(g)$. Then

- (A) $f = g = 0$
- (B) $\ker(g) = \ker(f)$
- (C) $\frac{\ker(g)}{\ker(f)}$ is isomorphic to \mathbb{R}
- (D) none of the above holds.

2. The dimension of the vector space consisting of all linear transformations from \mathbb{R}^5 to \mathbb{R} is

- (A) 1 (B) 0 (C) 5 (D) 6

3. The quotient ring $\frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$ is

- (A) a finite field
- (B) not an integral domain
- (C) an infinite field
- (D) an integral domain but not a field.

4. Find a true statement below:

- (A) $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}(\sqrt{3})$ as fields
- (B) $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}(\sqrt{3})$ as vector spaces over \mathbb{Q}
- (C) $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 2$
- (D) $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{6})$.

5. Let $f : \mathbb{Z}_6 \rightarrow \mathbb{Z}$ be a non-zero group homomorphism. Then

- (A) $f = 0$
- (B) f is surjective
- (C) f is injective
- (D) none of the above holds.

6. Let $A = \{\frac{1}{n} \sin(\frac{1}{n}) \mid n \in \mathbb{N}\}$. The find a true statement from below:

- (A) A has one limit point and it is 1
- (B) A has one limit point and it is -1
- (C) A has one limit point and it is 0
- (D) A has three limit points and they are 0, 1, -1 .

20. The point in the plane $x - y - z = 0$ and the point $(4, -1, 1)$ is

- (A) $(2, 1, 1)$ (B) $(1, 0, 1)$ (C) $(0, 4, -1)$

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7. Let $A = \{1, 2\} \subset \mathbb{R}$. Let $f(x) = \inf\{|x - a| \mid a \in A\}$ ($x \in \mathbb{R}$). Then

- (A) f is not a continuous function
- (B) f is differentiable on $\mathbb{R} \setminus \{1, 2\}$
- (C) f is differentiable on $\mathbb{R} \setminus \{3/2\}$
- (D) f is differentiable on $\mathbb{R} \setminus \{1, 3/2, 2\}$.

8. Let $\mathcal{F} = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid |f(x) - f(y)| \leq C|x - y| \forall x, y \in \mathbb{R} \text{ for some constant } C\}$.

Find a true statement from below:

- (A) If $f \in \mathcal{F}$, then f is uniformly continuous on \mathbb{R}
- (B) If $f \in \mathcal{F}$, then f is differentiable on \mathbb{R}
- (C) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, then $f \in \mathcal{F}$
- (D) none of the above holds.

9. Find a true statement from below:

- (A) $n \log(1 + \frac{1}{n^2}) \rightarrow 1$ as $n \rightarrow \infty$
- (B) $n \log(1 + \frac{1}{n+1}) \rightarrow 1$ as $n \rightarrow \infty$
- (C) $(n + 1) \log(1 + \frac{1}{n}) \rightarrow 0$ as $n \rightarrow \infty$
- (D) $n^2 \log(1 + \frac{1}{n^2}) \rightarrow 0$ as $n \rightarrow \infty$.

10. Find the norm below with respect to which the space $C([0, 1])$ of continuous real valued functions on $[0, 1]$ is complete:

- (A) $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$
- (B) $\|f\|_2 = (\int_0^1 f(x)^2 dx)^{1/2}$
- (C) $\|f\|_1 = \int_0^1 |f(x)| dx$
- (D) none of the above.

11. Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a continuous function. In which case below imply that $f = 0$?

- (A) $\int_0^\pi x^n f(x) dx = 0 \forall n \in \mathbb{N} \cup \{0\}$
- (B) $0 = \int_0^\pi \cos(nx) f(x) dx \forall n \in \mathbb{N} \cup \{0\}$
- (C) $\int_0^\pi \sin(nx) f(x) dx = 0 \forall n \in \mathbb{N} \cup \{0\}$
- (D) $0 = \int_0^\pi (\cos(nx) + \sin(nx)) f(x) dx \forall n \in \mathbb{N} \cup \{0\}$.

12. Find a true statement below:

- (A) $f(z) = \cos z$ ($z \in \mathbb{C}$) is bounded function.
- (B) There is an non-constant entire function with $f(\mathbb{C}) = \mathbb{R}$.
- (C) There exists a bounded non-constant entire function.
- (D) If f, g are entire functions such that $f(iy) = g(iy) \forall 0 < y < 1$, then $f = g$.

13. Which function below is uniformly continuous?

- (A) $f(x) = 1/x$ ($0 < x < 1$)
- (B) $f(x) = x^3$ ($x \in \mathbb{R}$)
- (C) $f(x) = \sin^2(x)$ ($x \in \mathbb{R}$)
- (D) $f(x) = \sin(1/x)$ ($0 < x < 1$)

14. Let $A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. Find a set below which is homeomorphic to A .

- (A) $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$
- (B) $\{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$
- (C) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
- (D) $\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$

15. Let

$$f(z) = \frac{e^z + 1}{e^z - 1}$$

$\forall z \in \mathbb{C}$. Then $z = 0$ is

- (A) a removable singularity of f
- (B) a pole of f of order 2
- (C) an essential singularity of f
- (D) a pole of f of order 1.

16. Let $\gamma : [0, \pi] \rightarrow \mathbb{C}$ be the contour defined by $\gamma(s) = 2e^{is} \forall s \in [0, \pi]$.

Then $\int_{\gamma} \frac{z+1}{z} dz =$

- (A) $i\pi$
- (B) $3 + i\pi$
- (C) $-4 + i\pi$
- (D) 2π

17. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $f : D \rightarrow \mathbb{C}$ be an analytic function such that $f(0) = 0$. Define $g(z) = \frac{f(z)}{z} \forall z \in D \setminus \{0\}$ and $g(0) = f'(0)$.

Then find the correct statement from below:

- (A) g is not continuous at $z = 0$
- (B) $z = 0$ is a removable singularity of g
- (C) g has a pole at $z = 0$
- (D) g is a meromorphic function.

18. Find a compact set below:

- (A) $\{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$
- (B) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{1 \leq i \leq n} x_i^2 = 1\}$
- (C) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{1 \leq i \leq n} x_i = 1\}$
- (D) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \dots x_n = 1\}$.

19. Find a true statement below:

- (A) If A is a dense subset of a topological space X , then $X \setminus A$ is nowhere dense in X .
- (B) If A is no-where dense subset of a topological space X , then $X \setminus A$ is dense in X .
- (C) $\mathbb{R} \times \{0\}$ is a dense subset of $\mathbb{R} \times \mathbb{R}$
- (D) $\{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is a dense subset of $\mathbb{R} \times \mathbb{R}$.

20. The point in the plane ' $x - y - z = 0$ ' in \mathbb{R}^3 which is nearest to the point $(4, -1, 1)$ is

- (A) $(2, 1, 1)$
- (B) $(1, 0, 1)$
- (C) $(0, 4, -4)$
- (D) $(5, 2, 3)$.

Section II

Answer any three questions.

[30]

1. Prove that any non-constant analytic map $f : \mathbb{C} \rightarrow \mathbb{C}$ is an open map.
2. Prove that any compact subset of a Hausdorff topological space X is a closed subset of X .
3. Give an example of a sequence of real valued continuous functions $(f_n)_{n \in \mathbb{N}}$ defined on $[0, 1]$ satisfying the following properties:

- (a) $\int_0^1 f_n(x) dx = 1 \forall n \in \mathbb{N}$ and
- (b) $f_n(x) \rightarrow 0$ as $n \rightarrow \infty \forall x \in [0, 1]$.

4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map such that the minimum polynomial $f(X)$ of T is product of k many distinct linear factors for some $1 \leq k \leq n$. Then prove that T is a daigonalisable linear map.
5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function. Fix $p \in \mathbb{R}^n$. Prove that there exist constants $r > 0$ and $M > 0$ such that

$$|f(x) - f(y)| \leq M\|x - y\|$$

for all $x, y \in \mathbb{R}^n$ satisfying $\|x - p\| < r, \|y - p\| < r$.

Section III

Answer any two questions.

[30]

1. Let G be a finite group. For $x \in G$, define its conjugacy class in G . Derive the conjugacy class equation for G . Deduce that a group of order p^2 is an abelian group where $p \in \mathbb{N}$ is any prime.
2. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a Lebesgue measurable function. Prove that there exists a sequence of simple functions $(s_n)_{n \in \mathbb{N}}$ satisfying the following properties:
 - (a) $0 \leq s_n(x) \leq s_{n+1}(x) \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$ and
 - (b) $s_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for each $x \in \mathbb{R}$.
3. Show that a complex polynomial of degree $n \in \mathbb{N}$ has exactly n many zeroes in \mathbb{C} .
4. Prove that ant isometry of \mathbb{R}^n is composite of at most $n + 1$ many reflections.
